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A HARNACK INEQUALITY FOR FRACTIONAL LAPLACE EQUATIONS WITH LOWER ORDER TERMS

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ABSTRACT. We establish a Harnack inequality of fractional Laplace equations without imposing sign condition on the coefficient of zero order term via the Moser's iteration and John-Nirenberg inequality.

1. INTRODUCTION

This note is devoted to a Harnack inequality of Laplace equations without imposing sign condition on the coefficient of zero order terms.

The fractional Laplacians $(-\Delta)^\sigma$, $0<\sigma<1$, which are the infinitesimal generators in stable Lévy stable processes, are given by the Fourier transform $\mathcal{F}$ as follows: for $u \in H^\sigma(\mathbb{R}^n)$, $n \geq 2$,

$$\mathcal{F}((-\Delta)^\sigma u)(\xi) := |\xi|^{2\sigma}\mathcal{F}(u(\xi))\ \xi \in \mathbb{R}^n. \tag{1.1}$$

Caffarelli and Silvestre [3] introduced fractional extension $v \in D^{1,2}_\sigma(\mathbb{R}^{n+1}_+)$ of $v(x,0)=u(x)$ satisfying

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla v(x,t)|^2\, t^{1-2\sigma} dt dx = c_\sigma \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\mathcal{F}(u)(\xi)|^2\, d\xi, \tag{1.2}$$

where $c_\sigma^{-1} = 2^{-1}(4\pi)^{2\sigma}\Gamma(2-2\sigma)$. Then the factional Laplacians are realized by the Dirichlet-Neumann map of v

$$(-\Delta)^\sigma u(x) = -c_\sigma \lim_{t\to 0} t^{1-2\alpha} v_t. \tag{1.3}$$

Let $B_r \subset \mathbb{R}^n$ be the ball centered at origin with radius r. Our main result is

Theorem 1.1. *Let $u \in H^\sigma(\mathbb{R}^n)$ be nonnegative in $\mathbb{R}^n$ and $C^2(B_1)\cap C^1(\overline{B_1})$. Suppose that $u(x)$ satisfies*

$$(-\Delta)^\sigma u(x) = a(x)u(x) + b(x) \quad \textit{in } B_1, \tag{1.4}$$

where $a(x), b(x) \in L^\infty(B_1)$. Then

$$\sup_{B_{1/2}} u \leq C(\inf_{B_{1/2}} u + \|b\|_{L^\infty(B_1)}),$$

where $C>0$ depends only on $n, \sigma, \|a\|_{L^\infty(B_1)}$.

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To prove it, we establish a Harnack inequality for the equivalent problem as follow.

Let $X = (x,t) \in \mathbb{R}^{n+1}$, $Q_R = B_R \times (0,R) \subset \mathbb{R}^{n+1}$ and $\partial' Q_R = B_R \times \{0\}$. Define

$$H(t^{1-2\sigma}, Q_R) := \left\{ U \in H^1(Q_R) : \int_{Q_R} t^{1-2\sigma}(U^2 + |\nabla U|^2)\,\mathrm{d}X < \infty \right\}.$$

Theorem 1.2. *Let $U \in H(t^{1-2\sigma}, Q_1)$ be nonnegative solution $C^2(Q_1) \cap C^1(\overline{Q_1})$ of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U(X)) = 0 & \text{in } Q_1 \\ -\lim\limits_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t) = a(x)U(x,0) + b(x) & \text{on } \partial' Q_1. \end{cases} \tag{1.5}$$

Suppose $a, b \in L^\infty(B_1)$. Then

$$\sup_{\overline{Q}_{1/2}} U \le C(\inf_{\overline{Q}_{1/2}} U + \|b\|_{L^\infty(B_1)}),$$

where $C > 0$ depends only on n, σ and $\|a\|_{L^\infty(B_1)}$.

The main feature is that we do not assume the sign condition of $a(x)$. Previously, in the case $a(x) \equiv 0$, Bass and Levin [1] establish the Harnack inequality for nonnegative functions of a class of symmetric stable processes that are harmonic with respect to these processes, see also [4] by Chen and Song. The analytic method was given by Caffarelli and Silvestre [3], by employing the fractional extension of fractional harmonic functions.

We here establish the Harnack inequality as in Theorem 1.2 by the Moser iteration. The proof bases on the properties of the weighted Sobolev space developed by Fabes, Kenig and Serapioni [5] and the John-Nirenberg inequality in A_2 weighted BMO space obtained by Muchenhoupt and Wheeden [10].

If $\sigma = \frac{1}{2}$, the result is due to Han and Li [7]. After we complete our manuscript, we observe that Theorem 1.2, the Harnack inequality for $b \equiv 0$, has been shown recently by Cabre and Sire [2] through making even extension and using the result of Fabes, Kenig and Serapioni [5]. But our proof has independent interest.

On the other hand, since the fractional Laplacian is a nonlocal operator, the condition $u \ge 0$ in $\mathbb{R}^n$ cannot be relaxed to $u \ge 0$ in B_1. In fact, we need all information in the complement of B_1. For example, see an counterexample of the case $a \equiv b \equiv 0$ in [9] by Kassmann. By the Dirichlet-Neumann map, we transform (1.4) to the local problem in $\mathbb{R}^{n+1}_+$, which grantees the identity (1.2). The nonnegative assumption of u implies that its fractional extension v is nonnegative in the half space $\mathbb{R}^{n+1}_+$. Thus, $v \ge 0$ in all of cubes $Q_R, R > 0$. Therefore, we can obtain the desired Harnack inequality by studying the local version (1.5).

The paper is organized as follows. In Section 2, we demonstrate some properties in the weighted Sobolev spaces. The proofs of Theorem 1.1, 1.2 are given in Section 3.

2. Preliminaries

In this section, we shall present some important weighted inequalities.

Denote $Q_R = B_R \times (0,R) \subset \mathbb{R}^{n+1}$, $\partial' Q_R = B_R \times \{0\}$ and $\partial'' Q_R = \partial Q_R \setminus \partial' Q_R$. We use capital letters like $X = (x,t), Y = (y,s)$ to represent points in $\mathbb{R}^{n+1}$.

Let us recall the definition of A_2 class.

Definition 2.1. *Let $\omega(X)$ be a nonnegative measurable function in $\mathbb{R}^{n+1}$. We say ω being of the class A_2 if there exists a constant C_ω such that for any ball $B \subset \mathbb{R}^{n+1}$*

$$\left(\frac{1}{|B|}\int_B \omega(X)\,\mathrm{d}X\right)\left(\frac{1}{|B|}\int_B \omega^{-1}(X)\,\mathrm{d}X\right) \le C_\omega,$$

where $|\cdot|$ is the Lebesgue measure.

Lemma 2.1. *Let $f(X) \in C_c^1(Q_R \cup \partial' Q_R)$ and $\omega(X) \in A_2$. Then there exist constants C and $\delta > 0$ depending only n and C_ω such that for any $1 \le k \le \frac{n+1}{n} + \delta$*

$$\left(\frac{1}{\omega(Q_R)}\int_{Q_R} |f|^{2k}\omega\,\mathrm{d}X\right)^{1/2k} \le CR\left(\frac{1}{\omega(Q_R)}\int_{Q_R} |\nabla f|^2\omega\,\mathrm{d}X\right)^{1/2}, \tag{2.1}$$

where $\omega(Q_R) = \int_{Q_R}\omega(X)\,\mathrm{d}X$.

Proof. The proof of this Lemma is similar to that of Theorem 1.2 in [5]. The following inequality is the only thing we need to show.

$$|f(X)| \le \frac{2}{\omega_n}\int_{Q_R}\frac{|\nabla f(Y)|}{|X-Y|^n}\,\mathrm{d}Y, \quad \text{for any } X \in Q_R, \tag{2.2}$$

where ω_n is the area of the sphere $\mathbb{S}^n$.

Extend f to be zero outside Q_R. Let $X \in Q_R$, then (2.2) follows from

$$f(X) = \frac{2}{\omega_n}\int_{\mathbb{R}_-^{n+1}}\frac{\nabla f(X-Y)\cdot Y}{|Y|^{n+1}}\,\mathrm{d}Y. \tag{2.3}$$

Since $X - Y \in \mathbb{R}_+^{n+1}$, $\nabla f(X-Y)$ makes sense. Let $\xi \in \mathbb{S}_-^n$, the south half sphere. For $t > 0$, note that

$$f(X) = \int_0^\infty -\frac{\partial}{\partial t} f(X-\xi t)\,\mathrm{d}t = \int_0^\infty \nabla f(X-\xi t)\cdot\xi\,\mathrm{d}t.$$

We integrate the above over ξ ranging on the south half sphere. This gives

$$f(X) = \frac{2}{\omega_n}\int_{\xi\in\mathbb{S}_-^n}\int_0^\infty \nabla f(X-\xi t)\cdot\xi\,\mathrm{d}t\mathrm{d}\xi.$$

Identity (2.3) follows from coordinate changing. □

Next we quote the following weighted Poincaré inequality which can be found in [5].

Lemma 2.2. *Let $f \in C^1(Q_R)$, then any $1 \le k \le \frac{n}{n-1} + \delta$, we have*

$$\left(\frac{1}{\omega(Q_R)}\int_{Q_R} |f - f_{R,\omega}|^{2k}\omega\,\mathrm{d}X\right)^{1/2k} \le CR\left(\frac{1}{\omega(Q_R)}\int_{Q_R} |\nabla f|^2\omega\,\mathrm{d}X\right)^{1/2},$$

where $f_{R,\omega} = \frac{1}{\omega(Q_R)}\int_{Q_R} f\omega$.

Finally, we prove the following trace embedding result.

Lemma 2.3. *Let $f(X) \in C_c^1(Q_R \cup \partial' Q_R)$ and $\alpha \in (-1, 1)$. Then there exists a positive constant δ depending only on α such that*

$$\int_{\partial' Q_R} |f|^2 \le \varepsilon\int_{Q_R} |\nabla f|^2 t^\alpha + \frac{C(R)}{\varepsilon^\delta}\int_{Q_R} |f|^2 t^\alpha, \tag{2.4}$$

for any $\varepsilon > 0$.

Proof. For any $1 < p < \infty$, we have

$$\begin{aligned}\int_{\partial' Q_R} |f|^p &= -\int_{Q_R} \partial_t |f|^p = -\int_{Q_R} p|f|^{p-1}\mathrm{sgn} f \partial_t f \\ &\le \varepsilon \int_{Q_R} |\nabla f|^p + C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} |f|^p.\end{aligned} \tag{2.5}$$

Next, we claim for $0 < \alpha < 1$ and any $\lambda > -1$

$$\int_{Q_R} |f|^2 t^\lambda \le C(\lambda, \alpha) \int_{Q_R} |\nabla f|^2 t^\alpha. \tag{2.6}$$

In fact, by the Hölder inequality

$$\begin{aligned}f^2(x,t) = (\int_t^R \partial_t f(x,s)\,\mathrm{d}s)^2 &\le \int_t^R s^{-\alpha}\,\mathrm{d}s \int_t^R |\partial_t f|^2 s^\alpha\,\mathrm{d}s \\ &\le \frac{C}{1-\alpha}\int_0^R |\nabla f(x,s)|^2 s^\alpha\,\mathrm{d}s.\end{aligned}$$

Multiplying the above by t^λ and integrating over Q_R, we obtain

$$\begin{aligned}\int_{Q_R} t^\lambda f^2 &\le C\int_0^R t^\lambda\,\mathrm{d}t \int_{B_R}\int_0^R |\nabla f(x,s)|^2 s^\alpha\,\mathrm{d}s\,\mathrm{d}x \\ &\le C\int_{Q_R} |\nabla f(x,s)|^2 s^\alpha,\end{aligned}$$

so (2.6) follows.

We are going to prove (2.4). Let $p \in (1, \frac{2}{1+\alpha})$. It follows from (2.5) and the Hölder inequality that

$$\begin{aligned}\int_{\partial' Q_R} |f|^2 &= \int_{\partial' Q_R} (|f|^{\frac{2}{p}})^p \\ &\le \varepsilon \int_{Q_R} |\nabla f^{\frac{2}{p}}|^p + C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} |f|^2 \\ &= \varepsilon(\frac{2}{p})^p \int_{Q_R} |f|^{2-p} t^{-\frac{p\alpha}{2}} |\nabla f|^p t^{\frac{p\alpha}{2}} + C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} |f| t^{-\frac{\alpha}{2}} |f| t^{\frac{\alpha}{2}} \\ &\le \varepsilon(\frac{2}{p})^p \left(\int_{Q_R} |f|^2 t^{-\frac{p\alpha}{2-p}}\right)^{\frac{2-p}{2}} \left(\int_{Q_R} |\nabla f|^2 t^\alpha\right)^{\frac{p}{2}} \\ &\quad + C\varepsilon^{-\frac{1}{p-1}} \int_{Q_R} \{\varepsilon^{1+\frac{1}{p-1}} |f|^2 t^{-\alpha} + \varepsilon^{-1-\frac{1}{p-1}} |f|^2 t^\alpha\} \\ &\le \varepsilon C \int_{Q_R} |\nabla f|^2 t^\alpha + \frac{C}{\varepsilon^{1+\frac{2}{p-1}}} \int_{Q_R} |f|^2 t^\alpha,\end{aligned}$$

where we used (2.6) for $\lambda = -\frac{p\alpha}{2-p} > -1$ and $\lambda = -\alpha > -1$ in the last inequality. Therefore, we complete the proof. $\square$

3. Proof of Theorem 1.1

In this section, we will prove the main results by making use of the Moser's iteration. For $p \in (0,\infty)$ denote

$$\|U\|_{L^p(t^{1-2\sigma},Q_R)} := \left(\int_{Q_R} t^{1-2\sigma}U^p\right)^{\frac{1}{p}}.$$

Proposition 3.1. *Let $U(X) \in H(t^{1-2\sigma}, Q_1)$ be a weak solution of*

$$\begin{cases} \mathrm{div}(t^{1-2\sigma}\nabla U(X)) \geq 0 & \text{in } Q_1 \\ -\lim_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t) \leq a(x)U(x,0) + b(x) & \text{on } \partial' Q_1. \end{cases} \tag{3.1}$$

Then

$$\sup_{Q_{1/2}} U^+ \leq C(\|U^+\|_{L^2(t^{1-2\sigma},Q_R)} + \|b\|_{L^\infty(B_1)}),$$

where $U^+ = \max\{0,U\}$, and $C > 0$ depends only on $n, \sigma, \|a\|_{L^\infty(B_1)}$.

Proof. Let $k, m > 0$ be some constants. Set $\overline{U} = U^+ + k$ and

$$\overline{U}_m = \begin{cases} \overline{U} & \text{if } U < m, \\ k+m & \text{if } U \geq m. \end{cases}$$

Consider the test function

$$\phi = \eta^2(\overline{U}_m^\beta \overline{U} - k^{\beta+1}) \in H(t^{1-2\sigma}, Q_1),$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_c^1(Q_1 \cup \partial' Q_1)$. Clearly, $\nabla \overline{U}_m = 0$ in $\{U < 0\}$ and $\{U \geq m\}$. A direct calculation yields

$$\begin{aligned} \nabla\phi &= \beta\eta^2\overline{U}_m^{\beta-1}\nabla\overline{U}_m\overline{u} + \eta^2\overline{U}_m^\beta\nabla\overline{U} + 2\eta\nabla\eta(\overline{U}_m^\beta\overline{U} - k^{\beta+1}) \\ &= \eta^2\overline{U}_m^\beta(\beta\nabla\overline{U}_m + \nabla\overline{U}) + +2\eta\nabla\eta(\overline{U}_m^\beta\overline{U} - k^{\beta+1}). \end{aligned}$$

Multiplying (3.1) by ϕ and integrating by parts, we have

(3.2)

$$\begin{aligned} 0 &\leq -\int_{Q_1} t^{1-2\sigma}\nabla U\nabla\phi + \int_{\partial' Q_1} a(x)U\phi + b(x)\phi \\ &= -\int_{Q_1} t^{1-2\sigma}\eta^2\overline{U}_m^\beta(\beta|\nabla\overline{U}_m|^2 + |\nabla\overline{U}|^2) - 2\int_{Q_1} t^{1-2\sigma}\eta(\overline{U}_m^\beta\overline{U} - k^{\beta+1})\nabla\eta\nabla\overline{U} \\ &\quad + \int_{\partial' Q_1} a(x)U\eta^2(\overline{U}_m^\beta\overline{U} - k^{\beta+1}) + b(x)\eta^2(\overline{U}_m^\beta\overline{U} - k^{\beta+1}) \\ &\leq -\frac{1}{2}\int_{Q_1} t^{1-2\sigma}\eta^2\overline{U}_m^\beta(\beta|\nabla\overline{U}_m|^2 + |\nabla\overline{U}|^2) + 4\int_{Q_1} t^{1-2\sigma}\overline{U}_m^\beta\overline{U}^2|\nabla\eta|^2 \\ &\quad + \int_{\partial' Q_1} |a(x)|\eta^2\overline{U}_m^\beta\overline{U}^2 + |b(x)|\eta^2\overline{U}_m^\beta\overline{U}, \end{aligned}$$

where we used the Cauchy inequality and the fact $\overline{U}_m^\beta\overline{U} - k^{\beta+1} < \overline{U}_m^\beta\overline{U}$. Choosing $k = \|b\|_{L^\infty(B_1)}$ if b is not identically zero. Otherwise choose an arbitrary $k > 0$ and

eventually let $k\to 0$. Then we see that $|b(x)|\eta^2\overline{U}_m^\beta\overline{U}\le\eta^2\overline{U}_m^\beta\overline{U}^2$. Hence (3.2) gives

$$\int_{Q_1}t^{1-2\sigma}\eta^2\overline{U}_m^\beta(\beta|\nabla\overline{U}_m|^2+|\nabla\overline{U}|^2)$$
$$\le 8\int_{Q_1}t^{1-2\sigma}\overline{U}_m^\beta\overline{U}^2|\nabla\eta|^2+2(\|a\|_{L^\infty(B_1)}+1)\int_{\partial'Q_1}\eta^2\overline{U}_m^\beta\overline{U}^2.$$

Set $W=\overline{U}_m^{\frac{\beta}{2}}\overline{U}$. Then

$$|\nabla W|^2\le(1+\beta)(\beta\overline{U}_m^\beta|\nabla\overline{U}_m|^2+\overline{U}_m^\beta|\nabla\overline{U}|^2).$$

Therefore, we have

$$\int_{Q_1}t^{1-2\sigma}\eta^2|\nabla W|^2\le C(1+\beta)\left\{\int_{Q_1}t^{1-2\sigma}W^2|\nabla\eta|^2+\int_{\partial'Q_1}\eta^2W^2\right\},$$

or

$$\int_{Q_1}t^{1-2\sigma}|\nabla(\eta W)|^2\le C(1+\beta)\left\{\int_{Q_1}t^{1-2\sigma}W^2|\nabla\eta|^2+\int_{\partial'Q_1}\eta^2W^2\right\}.$$

By Lemma 2.3,

$$C(1+\beta)\int_{\partial'Q_1}\eta^2W^2\le\frac{1}{2}\int_{Q_1}t^{1-2\sigma}|\nabla(\eta W)|^2+C(1+\beta)^\delta\int_{Q_1}t^{1-2\sigma}\eta^2W^2$$

for some $\delta>1$ depending on n,σ. It follows that

$$\int_{Q_1}t^{1-2\sigma}|\nabla(\eta W)|^2\le C(1+\beta)^\delta\int_{Q_1}t^{1-2\sigma}(\eta^2+|\nabla\eta|^2)W^2.$$

By the Sobolev inequality, see Lemma 2.2, we obtain

$$\left(\int_{Q_1}t^{1-2\sigma}|\eta W|^{2\chi}\right)^{\frac{1}{\chi}}\le C(1+\beta)^\delta\int_{Q_1}t^{1-2\sigma}(\eta^2+|\nabla\eta|^2)W^2,$$

where $\chi=\frac{n+1}{n}>1$. For any $0<r<R\le 1$, consider an $\eta\in C_c(Q_1\cup\partial'Q_1)$ with $\eta=1$ in Q_r and $|\nabla\eta|\le 2/(R-r)$. Thus we have

$$\left(\int_{Q_r}t^{1-2\sigma}W^{2\chi}\right)^{\frac{1}{\chi}}\le C\frac{(1+\beta)^\delta}{(R-r)^2}\int_{Q_R}t^{1-2\sigma}W^2.$$

or, by the definition of W,

$$\left(\int_{Q_r}t^{1-2\sigma}\overline{U}_m^{\beta\chi}\overline{U}^{2\chi}\right)^{\frac{1}{\chi}}\le C\frac{(1+\beta)^\delta}{(R-r)^2}\int_{Q_R}t^{1-2\sigma}\overline{U}_m^\beta\overline{U}^2.$$

Noting that $\overline{U}_m\le\overline{U}$, we get

$$\left(\int_{Q_r}t^{1-2\sigma}\overline{U}_m^{\gamma\chi}\right)^{\frac{1}{\chi}}\le C\frac{(1+\beta)^\delta}{(R-r)^2}\int_{Q_R}t^{1-2\sigma}\overline{U}^\gamma$$

provided the integral in the right hand side is bounded. By letting $m\to\infty$, we conclude that

$$\|\overline{U}\|_{L^{\gamma\chi}(t^{1-2\sigma},Q_r)}\le\left(C\frac{(1+\beta)^\delta}{(R-r)^2}\right)^{\frac{1}{\gamma}}\|\overline{U}\|_{L^\gamma(t^{1-2\sigma},Q_R)},$$

where $C>0$ is a constant depending only $n,\sigma,\|a\|_{L^\infty(B_1)}$. As in standard Moser iterating procedure, we then arrive at

$$\sup_{Q_{1/2}}\overline{U}\le C\|\overline{U}\|_{L^2(t^{1-2\sigma},Q_1)}$$

or

$$\sup_{Q_{1/2}}U^+\le C(\|U^+\|_{L^2(t^{1-2\sigma},Q_1)}+k).$$

Recalling the definition of k, we complete the proof. □

The next lemma is so called weak Harnack inequality.

Proposition 3.2. *Let $U(X)\in H(t^{1-2\sigma},Q_1)$ be a nonnegative weak solution of*

$$\begin{cases}\operatorname{div}(t^{1-2\sigma}\nabla U(X))\le 0 & \text{in } Q_1\\ -\lim\limits_{t\to0^+}t^{1-2\sigma}\partial_tU(x,t)\ge a(x)U(x,0)+b(x) & \text{on } \partial'Q_1.\end{cases}\tag{3.3}$$

Then for some $p>0$ and any $0<\theta<\tau<1$ we have

$$\inf_{Q_\theta}U+\|b\|_{L^\infty(Q_1)}\ge C\|U\|_{L^p(t^{1-2\sigma},Q_\tau)},$$

where $C>0$ depends only on $n,\sigma,\theta,\tau,\|a\|_{L^\infty(Q_1)}$.

Proof. Set $\overline{U}=U+k>0$, for some positive k to be determined and $V=\overline{U}^{-1}$. Let Φ be any nonnegative function in $H(t^{1-2\sigma},Q_1)$ with compact support in $Q_1\cup\partial'Q_1$. Multiplying both sides of first inequality of (3.3) by $\overline{U}^{-2}\Phi$ and integrating by parts, we obtain

$$0\ge-\int_{Q_1}t^{1-2\sigma}\frac{\nabla U\nabla\Phi}{\overline{U}^2}+2\int_{Q_1}t^{1-2\sigma}\nabla U\nabla\overline{U}\frac{\Phi}{\overline{U}^3}+\int_{\partial'Q_1}(aU+b)\overline{U}^{-2}\Phi.$$

Note that $\nabla U=\nabla\overline{U}$ and $\nabla V=-\overline{U}^2\nabla\overline{U}$. Therefore, we have

$$\int_{Q_1}t^{1-2\sigma}\nabla V\nabla\Phi+\int_{\partial'Q_1}\tilde{a}V\Phi\le0,$$

where

$$\tilde{a}=\frac{aU+b}{\overline{U}}.$$

Choose $k=\|b\|_{L^\infty(Q_1)}$ if b is not identical zero. Otherwise, choose an arbitrary $k>0$ and eventually let it tend to zero. Note that $\|\tilde{a}\|_{L^\infty(Q_1)}\le\|a\|_{L^\infty(Q_1)}+1$. Therefore Proposition 3.1 implies that for any $\tau\in(\theta,1)$ and any $p>0$

$$\sup_{Q_\theta}V\le C\|V\|_{L^p(t^{1-2\sigma},Q_\tau)},$$

or,

$$\begin{aligned}\inf_{Q_\theta}\overline{U}&\ge C\left(\int_{Q_\tau}t^{1-2\sigma}\overline{U}^{-p}\right)^{-\frac1p}\\&=C\left(\int_{Q_\tau}t^{1-2\sigma}\overline{U}^{-p}\int_{Q_\tau}t^{1-2\sigma}\overline{U}^{p}\right)^{-\frac1p}\left(\int_{Q_\tau}t^{1-2\sigma}\overline{U}^{p}\right)^{\frac1p},\end{aligned}$$

where $C>0$ depends only on n,σ,p,θ,τ.

The next key point is to show that there exists some $p_0>0$ such that

$$\int_{Q_\tau}t^{1-2\sigma}\overline{U}^{-p_0}\int_{Q_\tau}t^{1-2\sigma}\overline{U}^{p_0}\le C,$$

where $C > 0$ depends only on n, σ, τ. We are going to show that for any $\tau < 1$

$$\int_{Q_\tau} e^{p_0|W|} \le C, \tag{3.4}$$

where $W = \log \overline{U} - (\log \overline{U})_{0,\tau}$. The idea is as usual. (3.4) will follows from John-Nirenberg type lemma (see [10]) if $W \in BMO(t^{1-2\sigma}\mathrm{d}X)$.

We first derive an equation for W. Multiplying both sides of first inequality of (3.3) by $\overline{U}^{-1}\Phi$ and integrating by parts, we obtain

$$\int_{Q_1} t^{1-2\sigma}|\nabla W|^2\Phi \le \int_{Q_1} t^{1-2\sigma}\nabla W\nabla\Phi + \int_{\partial' Q_1} \tilde{a}\Phi,$$

where

$$\tilde{a} = \frac{aU+b}{\overline{U}}.$$

Replace Φ by Φ^2. It follows from the Cauchy inequality and the Sobolev inequality that

$$\int_{Q_1} t^{1-2\sigma}|\nabla W|^2\Phi^2 \le C\int_{Q_1} t^{1-2\sigma}|\nabla\Phi|^2, \tag{3.5}$$

where $C > 0$ depends only on n, σ. Then for any $Q_{2r}(Y) \subset Q_1$, $Y \in \partial\mathbb{R}^{n+1}_+$, choose Φ with

$$\mathrm{supp}(\Phi) \subset Q_2r(Y) \cup \partial' Q_2r(Y), \quad \Phi = 1 \text{ in } Q_r(Y) \cup \partial' Q_r(Y), \quad |\nabla\Phi| \le \frac{C}{r}.$$

We have

$$\int_{Q_r(Y)} t^{1-2\sigma}|\nabla W|^2 \le \frac{C}{r^2}\int_{Q_r(Y)} t^{1-2\sigma}.$$

Hence the Poincaré inequality, Lemma 2.2, implies

$$\begin{aligned}
&\left(\int_{Q_r(Y)} t^{1-2\sigma}\right)^{-1}\int_{Q_r(Y)} t^{1-2\sigma}|W - W_{Y,r}| \\
&\qquad \le \left(\int_{Q_r(Y)} t^{1-2\sigma}\right)^{-1/2}\left(\int_{Q_r(Y)} t^{1-2\sigma}|W - W_{Y,r}|^2\right)^{1/2} \\
&\qquad \le r\left(\int_{Q_r(Y)} t^{1-2\sigma}\right)^{-1/2}\left(\int_{Q_r(Y)} t^{1-2\sigma}|\nabla W|^2\right)^{1/2} \\
&\qquad \le C.
\end{aligned}$$

For other $Y \in Q_1$, one can verify the above similarly. Therefore, we conclude that $W \in BMO(t^{1-2\sigma}, Q_1)$. □

Proof of Theorem 1.2. The proof follows from Proposition 3.1 and 3.2. □

Proof of Theorem 1.1. Since $u \ge 0$ in $\mathbb{R}^n$ be a solution of (1.4), there exists a nonnegative function $U(x,t) \in H(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ satisfying

$$\mathrm{div}(t^{1-2\sigma}\nabla U(x,t)) = 0 \text{ in } \mathbb{R}^{n+1}_+$$

and $U(x,0) = u(x)$. It follows from (1.3) that

$$\begin{aligned}
\lim_{t\to 0^+} t^{1-2\sigma}\partial_t U(x,t) &= -c_\sigma(-\Delta)^\sigma U(x,0) \\
&= -c_\sigma(a(x)U(x,0) + b(x)),
\end{aligned}$$

where we used $u \in C^2(B_1)$. Hence Theorem 1.1 immediately follows from Theorem 1.2. $\square$

Theorem 3.1. *Let* $0 < \sigma < 1$ *and* $B_R = B_R(0) \subset \mathbb{R}^n$, $n > 2\sigma$. *Suppose that* $a(x) \in L^\infty(\mathbb{R}^n)$, $0 \le u \in C(\mathbb{R}^n)$ *satisfies*

$$(-\Delta)^\sigma u(x) = a(x)u(x), \quad x \in B_R.$$

Then for $\delta > 0$, *there exists* $C(n,\sigma,\delta) > 0$ *such that*

$$\max_{\overline{B}_{R-\delta}} u \le C(n,\sigma,\delta) \min_{\overline{B}_{R-\delta}} u.$$

Proof. By rescaling, we can prove it from Theorem 1.1. See another proof in [2]. $\square$

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REFERENCES

[1] R.F. Bass and D.A. Levin: *Harnack inequalities for jump processes*, Potential Anal., **17** (2002) 375–388

[2] X. Cabre and Y. Sire: *Nonlinear equations for fractional laplacians I: regularity, maximum principles, and hamiltonian estimates*, preprint,

[3] L. Caffarelli and L. Silvestre: *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260

[4] Z.-Q. Chen and R. Song: *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. **312** (1998), 465-501.

[5] E. Fabes; C. Kenig and R. Serapioni: *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), 77–116

[6] Q. Han and F.-H. Lin: "Elliptic partial differential equations," American Mathematical Society, Providence, R.I., 1997

[7] Z.-C. Han and Y.Y. Li: *The Yamabe problem on manifolds with boundary: existence and compactness results*, Duke Math. J. **99** (1999), 489–542

[8] F. John and L. Nirenberg: *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961) 415–426

[9] M. Kassmann: *The classical Harnack inequality fails for non-local operators*, preprint

[10] B. Muckenhoupt and R. Wheeden: *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274

[11] E. Stein: "Singular integrals and differentiability properties of function," Princeton, 1970

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